The Fujita phenomenon in exterior domains under dynamical boundary conditions II*†

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Abstract

The Fujita phenomenon for nonlinear parabolic problems $\partial_t u = \Delta u + u^p$ in an exterior domain of $\mathbb{R}^N$ under dissipative dynamical boundary conditions $\sigma\partial_t u + \partial_\nu u = 0$ is investigated in the superlinear case. As in the case of Dirichlet boundary conditions (see Refs. [2] and [9]), it turns out that there exists a critical exponent $p = 1 + \frac{2}{N}$ such that blow-up of positive solutions always occurs for subcritical exponents, whereas in the supercritical case global existence can occur for small non-negative initial data.

Key words: Nonlinear parabolic problems; Dynamical boundary conditions; Global solutions.

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1 Introduction

Let Ω be an exterior domain of $\mathbb{R}^N$, that is to say a connected open set Ω such that $\overline{\Omega}^c$ is a bounded domain when $N \geq 2$, and in dimension one, Ω is the complement of a real closed interval. We always suppose that the boundary $\partial\Omega$ is of class $\mathcal{C}^2$. The outer normal unit vector field is denoted by $\nu : \partial\Omega \to \mathbb{R}^N$ and the outer normal derivative by ∂_ν. Let p be a real number with $p > 1$ and φ be a continuous function in $\overline{\Omega}$. Consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, \infty), \\ \mathcal{B}_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}\ . \end{cases} \tag{1}$$

The aim of this paper is to show that the well-known Fujita phenomenon in the case of $\Omega = \mathbb{R}^N$ (see Ref. [6]) and in the case of Dirichlet boundary conditions (see Refs. [2] and [9]) still holds for the dynamical boundary conditions. One can notice that dynamical boundary conditions $\mathcal{B}_\sigma(u) = 0$ with $\sigma \equiv 0$ correspond to the Neumann boundary conditions, which case has been discussed by Levine and Zhang [8]. It is already known, by Bandle, von Below and Reichel in [1], that for $p \in (1, 1 + \frac{2}{N})$, also for $p = 1 + \frac{2}{N}$ if $N \geq 3$, and for constant coefficient $\sigma \in [0, \infty)$, all positive solutions of (1) blow up in finite time. In addition, if the complement is star-shaped there exist global positive solutions of class $\mathcal{C}^1$ for $p > 1 + \frac{2}{N}$ by [1]. Our purpose is to show the existence of global positive solutions of Problem (1) for sufficiently small initial data in the supercritical case ($p > 1 + \frac{2}{N}$) for any exterior domain. Moreover our condition on σ is more general. Throughout, we shall assume the dissipativity condition

$$\sigma \geq 0 \text{ on } \partial\Omega \times (0, \infty) \tag{2}$$

and dealing with classical solutions

$$\sigma \in \mathcal{C}^1(\partial\Omega \times (0, \infty)). \tag{3}$$

The initial data is always supposed to be continuous, non-trivial, bounded, non-negative in $\overline{\Omega}$, and vanishing at infinity:

$$\varphi \in \mathcal{C}(\overline{\Omega}),\ 0 < \| \varphi \|_\infty < \infty,\ \varphi \geq 0,\ \lim_{\|x\|_2 \to \infty} \varphi(x) = 0. \tag{4}$$

In the case $\Omega = \mathbb{R}^N$, the boundary condition is dropped and the result is well known by the classical paper of Fujita [6]. Thus, we will suppose $\Omega \neq \mathbb{R}^N$.

2 Preliminaries

First, we give the definition of positive solution which is understood along this paper.

Definition 2.1 *A positive solution of Problem (1) is a positive function $u : (x,t) \mapsto u(x,t)$ of class $\mathcal{C}(\overline{\Omega} \times [0,T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$, satisfying*

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0,T), \\ \mathcal{B}_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = \varphi & \text{in } \overline{\Omega}\ , \end{cases}$$

where φ is a function, given in $\mathcal{C}(\overline{\Omega})$. The time $T \in [0,\infty]$ is the maximal existence time of the solution u. If $T = \infty$, the solution u is called global.

From [2], if $T < \infty$, u blows up in finite time, that is to say:

$$\lim_{t \nearrow T} \sup_{x \in \overline{\Omega}} u(x,t) = \infty.$$

Note that for initial data φ of class $\mathcal{C}^2(\overline{\Omega})$, the solution u is $\mathcal{C}^{2,1}(\overline{\Omega} \times [0,T))$, whereas $u \in \mathcal{C}(\overline{\Omega} \times [0,T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$ if φ is only continuous in $\overline{\Omega}$. Then, let us recall a standard procedure to construct solutions of Problem (1) in outer domains for uniformly bounded and continuous initial data φ. Let $B(0,R)$ be the ball centered at the origin of radius $R > 0$ such that $\overline{\Omega}^c \subset B(0,R)$. For any $n \in \mathbb{N}$, we set $B_n := B(0, R+n)$ and $\Omega_n := \Omega \cap B_n$. The boundary of Ω_n is

decomposed into two disjoint open sets:

$$\partial\Omega_n = \partial\Omega \dot{\cup} \partial B_n.$$

Define also an increasing sequence of initial data $(\varphi_n)_{n\in\mathbb{N}^*}$ such that

$$\begin{aligned} 0 \le \varphi_n \le \varphi \quad & \text{in} \quad \overline{\Omega}_n, \\ \varphi_n \equiv 0 \quad & \text{on} \quad \partial B_n, \\ \varphi_n = \varphi \quad & \text{in} \quad \overline{\Omega}_{n-1}, \end{aligned} \tag{5}$$

and consider the following problem with mixed boundary conditions

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega}_n \times (0,\infty), \\ \mathcal{B}_\sigma(u) := \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = 0 & \text{on } \partial B_n \times (0,\infty), \\ u(\cdot,0) = \varphi_n & \text{in } \overline{\Omega}_n\ . \end{cases} \tag{$P(n)$}$$

Let z be the maximal solution of

$$\begin{cases} \dot{z} = z^p, \\ z(0) = \| \varphi \|_\infty, \end{cases}$$

with maximal existence time $t_0 = \frac{1}{(p-1)\|\varphi\|_\infty^{p-1}}$. It is known from [4] that, for each $n \in \mathbb{N}^*$, Problem $(P(n))$ has a solution $u_n \in \mathcal{C}(\overline{\Omega}_n \times [0,T_n)) \cap \mathcal{C}^{2,1}(\overline{\Omega}_n \times (0,T_n))$, where T_n is the maximal existence time of u_n. Moreover by comparison principle from [3], we have, for any $n \in \mathbb{N}^*$, $0 \le u_n \le u_{n+1}$ and $u_n(\cdot,t) \le z(t)$ in $\overline{\Omega}_n$, so we have also $t_0 \le T_n$. Hence we obtain a sequence $(u_n)_{n\in\mathbb{N}^*}$ of functions in $\mathcal{C}(\overline{\Omega}_n \times [0,t_0)) \cap \mathcal{C}^{2,1}(\overline{\Omega}_n \times (0,t_0))$. Then, standard arguments based on a priori estimates for the heat equation imply $u_n \to u$ in the sense of $\mathcal{C}^{2,1}_{loc}(\overline{\Omega} \times (0,t_0))$ as $n \to \infty$, where u is a positive solution of Problem (1), see Refs. [1] and [7]. Moreover, since u_n vanishes on ∂B_n for each $n \in \mathbb{N}^*$, the solution u vanishes at

infinity:

$$\lim_{\|x\|_2 \to \infty} u(x,t) = 0 \ , \forall \ t \in (0,T) \ .$$

Note that t_0 is only a lower bound for the maximal existence time of solutions u_n and u, and it is possible that the times T_n and T are infinite. Indeed, results on blow-up for problems under dynamical boundary conditions from [5] can not be applied to the problems $(P(n))$ with mixed boundary conditions because their solutions u_n vanish on a part of the boundary.

3 Global existence in dimension $N \geq 3$

Throughout this section, we consider supercritical exponent p:

$$p > 1 + \frac{2}{N}.$$

Our technique will be to construct a function that bounds from above each solution u_n of Problem $(P(n))$. This will give us a sequence $(u_n)_{n\in\mathbb{N}^*}$ of global solutions in $\mathcal{C}(\overline{\Omega}_n \times [0,\infty)) \cap \mathcal{C}^{2,1}(\overline{\Omega}_n \times (0,\infty))$, thus the solution u of (1) must be global too. We will proceed by using the solution of the Neumann Problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0,\infty), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) = \psi & \text{in } \overline{\Omega} \ , \end{cases} \tag{6}$$

with ψ verifying (4). In [8], Levine and Zhang proved that Problem (6) admits global positive solutions for sufficiently small initial data. We show that the solution v of Problem (6) bounds from above the solution u of Problem (1) if the initial data are well ordered ($\varphi \leq \psi$) and if ψ satisfies the following hypotheses:

$$\psi \in \mathcal{C}^2(\overline{\Omega}) \ , \tag{7}$$

and for every $n \in \mathbb{N}^*$

$$\Delta\psi_n + \psi_n^p \geq 0 \text{ in } \overline{\Omega}_n \ , \tag{8}$$

where $(\psi_n)_{n\in\mathbb{N}^*}$ is the sequence of truncated initial data introduced in (5). We need a technical lemma, similar to Lemma 2.1 of [4].

Lemma 3.1 *Let ψ be a function satisfying (4), (7) and (8). For every $n \in \mathbb{N}^*$, the solution v_n of Problem $(P(n))$ under the Neumann boundary conditions and with the truncated initial data ψ_n verifies:*

$$\partial_t v_n \geq 0 \ \textit{in}\ \overline{\Omega}_n \times (0, T_{v_n}),$$

where T_{v_n} is the maximal existence time of v_n.

Proof : The function v_n is solution of the following problem

$$\begin{cases} \partial_t v_n = \Delta v_n + v_n^p & \text{in } \overline{\Omega}_n \times (0, T_{v_n}), \\ \partial_\nu v_n = 0 & \text{on } \partial\Omega \times (0, T_{v_n}), \\ v_n = 0 & \text{on } \partial B_n \times (0, T_{v_n}), \\ v_n(\cdot, 0) = \psi_n & \text{in } \overline{\Omega}_n \ , \end{cases}$$

in the bounded domain $\overline{\Omega}_n$. From (4) and from the strong maximum principle in [3], we claim

$$v_n > 0 \text{ on } (\Omega_n \cup \partial\Omega) \times (0, T_{v_n}).$$

Then, by regularity results from [7], we obtain $v_n \in \mathcal{C}^{2,2}(\overline{\Omega}_n \times (0, T_{v_n})$, and for $y = \partial_t v_n \in \mathcal{C}^{2,1}(\overline{\Omega}_n \times (0, T_{v_n})$ we have

$$\begin{cases} \partial_t y = \Delta y + p v_n^{p-1} y & \text{in } \overline{\Omega}_n \times (0, T_{v_n}), \\ \partial_\nu y = 0 & \text{on } \partial\Omega \times (0, T_{v_n}), \\ y = 0 & \text{on } \partial B_n \times (0, T_{v_n}), \end{cases}$$

and $y(\cdot, 0) \geq 0$ in $\overline{\Omega}_n$ thanks to (8). By the comparison principle in [3], we conclude: $y \geq 0$ in $\overline{\Omega}_n \times (0, T_{v_n})$.

$\blacksquare$

Lemma 3.2 *Let a coefficient σ verifying (2) and (3), two functions φ and ψ satisfying (4) and ψ with (7) and (8). If*

$$\varphi \leq \psi \ in \ \overline{\Omega}, \tag{9}$$

then Problem (1) with initial data φ admits a solution u verifying

$$u \leq v \ in \ \overline{\Omega} \times (0, T_v),$$

and

$$0 < T_v \leq T \leq \infty,$$

where v is solution of Problem (6) with initial data ψ, of maximal existence time T_v.

Proof : We consider the sequences of truncated solutions $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ respectively associated to the solutions u and v. Let $n \in \mathbb{N}^*$. First, we show that $v_n \leq v$ in $\overline{\Omega}_n \times [0, T_v)$. By construction (5), we have $\psi_n \leq \psi$ in $\overline{\Omega}_n$. Since v is a positive solution of Problem (6), it satisfies

$$\begin{cases} \partial_t v \geq \Delta v + v^p & \text{in } \overline{\Omega}_n \times (0, T_v), \\ \partial_\nu v \geq 0 & \text{on } \partial\Omega \times (0, T_v), \\ v \geq 0 & \text{on } \partial B_n \times (0, T_v), \\ v(\cdot, 0) \geq \psi_n & \text{in } \overline{\Omega}_n \ . \end{cases}$$

As v_n is a positive solution of $(P(n))$ under Neumann boundary conditions, we obtain from the comparison principle in [3]

$$v_n \leq v \text{ in } \overline{\Omega}_n \times [0, \tau), \tag{10}$$

for all $0 < \tau < \min\{\ T_{v_n},\ T_v\ \}$. We deduce $T_v \leq T_{v_n}$. Then, we show that $u_n \leq v_n$ in $\overline{\Omega}_n \times [0, T_{v_n})$. The previous lemma ensures that $\partial_t v_n \geq 0$. From (2), we obtain

$$\sigma \partial_t v_n + \partial_\nu v_n \geq 0 \text{ on } \partial\Omega \times (0, T_{v_n}).$$

Next, v_n is a positive solution of $(P(n))$, and thanks to (9), v_n verifies

$$\begin{cases} \partial_t v_n \geq \Delta v_n + v_n^p & \text{in } \overline{\Omega}_n \times (0, T_{v_n}), \\ \sigma \partial_t v_n + \partial_\nu v_n \geq 0 & \text{on } \partial\Omega \times (0, T_{v_n}), \\ v_n \geq 0 & \text{on } \partial B_n \times (0, T_{v_n}), \\ v_n(\cdot, 0) \geq \varphi_n & \text{in } \overline{\Omega}_n\ . \end{cases}$$

Again, by the comparison principle in [3] and by definition of u_n, we obtain

$$u_n \leq v_n \text{ in } \overline{\Omega}_n \times [0, \tau), \tag{11}$$

for all $0 < \tau < \min\{\ T_n,\ T_{v_n}\ \}$, and hence $T_{v_n} \leq T_n$. From equations (10) and (11), we have $T_v \leq T_n$ and $u_n \leq v$ in $\overline{\Omega}_n \times [0, T_v)$. Thus the solution u of Problem (1), obtained as the limit of the sequence $(u_n)_{n \in \mathbb{N}^*}$ with the procedure described in section 2, verifies $u \leq v$ in $\overline{\Omega} \times [0, T_v)$ and $T_v \leq T$.

■

Now, we just have to choose an initial data ψ_* sufficiently small such that Problem (6) admits a global positive solution (see Ref. [8]), and satisfying (7) and (8).

Theorem 3.3 *Under conditions (2), (3) and (4), Problem (1) admits global positive solutions for sufficiently small initial data. Moreover, some of these solutions vanish at infinity.*

Proof : An initial data φ verifying (4) and with $\varphi \leq \psi_*$ in $\overline{\Omega}$ allows us to conclude thanks to Lemma 3.2.

■

Remark 3.4 *One can notice that only the dissipativity and the regularity of the coefficient σ are needed. We are not obliged to impose any restriction like σ bounded or $\partial_t \sigma \equiv 0$. Moreover, the hypotheses (7) and (8) on the initial data ψ of Problem (6) are strictly technical and do not concern the initial data φ of Problem (1).*

4 Global existence in lower dimension

In this case, we can not use Levine and Zhang's result because it is proved only for dimension $N \geq 3$: they used some estimates for Green's functions, specific to dimension $N \geq 3$. We need an additional hypothesis on the coefficient σ. There exists a constant $\varsigma \in [0, \infty)$ such that

$$\forall (x,t) \in \partial\Omega \times [0, \infty) \ : \ \sigma(x,t) \leq \varsigma \ . \tag{12}$$

We begin with the case of dimension 2. Until now, Bandle - von Below - Reichel's lemma, concerning star-shaped domains, is the best result:

Lemma 4.1 [1], Lemma 28. *Suppose σ is a positive constant. If Ω^C is star-shaped with respect to the origin and if $\min\limits_{\partial\Omega} |x \cdot \nu(x)| \geq \sigma N$, then there exist positive global solutions of Problem (1), which vanish at infinity, for sufficiently small initial data.*

This allows us to deduce the following result for problems with mixed boundary conditions.

Corollary 4.2 *Suppose conditions (2), (3) and (12). Let $y \in \partial\Omega$. There exists a neighborhood N_y of y relatively open in $\partial\Omega$ such that the following*

parabolic problem with mixed boundary conditions

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, \infty), \\ \mathcal{B}_\sigma(u) = 0 & \text{on } N_y \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \setminus N_y \times (0, \infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}\ . \end{cases}$$

admits global positive solutions which vanish at infinity, for sufficiently small initial data φ satisfying (4).

Proof : Let μ be a vector in $\mathbb{R}^N$ such that the scalar product between the vector $(y + \mu)$ and the outer normal unit vector at y satisfies

$$(y + \mu) \cdot \nu(y) < -\varsigma N. \tag{13}$$

Then, as the mapping $(\partial\Omega \ni z \mapsto (z + \mu) \cdot \nu(z) \in \mathbb{R})$ is continuous, the above inequality remains true on an open neighborhood $N_y \subseteq \partial\Omega$ of y. We obtain the statement of the corollary by using the comparison principle from [3] and the function U defined on $\overline{\Omega} \times [0, \infty)$ by

$$U(x, t) = A(t + 1)^{-\gamma} \exp \frac{- \parallel x + \mu \parallel_2^2}{4(t + 1)},$$

with $A = \dfrac{1}{2}\Big(\dfrac{N}{2} - \dfrac{1}{p - 1}\Big)^{\frac{1}{p-1}}$ and $\gamma = \frac{1}{p-1}$. It is clear that $U \geq 0$, belongs to $\mathcal{C}^{2,1}(\overline{\Omega} \times [0, \infty))$ and satisfies:

$$\begin{aligned} \partial_t U &= \Big(\frac{-\gamma}{t + 1} + \frac{\parallel x + \mu \parallel_2^2}{4(t + 1)^2}\Big)U, \\ \Delta U &= \Big(\frac{-N}{2(t + 1)} + \frac{\parallel x + \mu \parallel_2^2}{4(t + 1)^2}\Big)U, \\ \partial_\nu U &= \Big(\frac{-(x + \mu) \cdot \nu(x)}{2(t + 1)}\Big)U. \end{aligned}$$

We have:

$$\partial_t U - \Delta U = \Big(\frac{-\gamma}{t+1} + \frac{N}{2(t+1)}\Big)U = \Big(\frac{-2\gamma + N}{2(t+1)}\Big)U \ ,$$

and by definition of the constants A and γ, we obtain

$$\partial_t U - \Delta U - U^p \geq 0 \text{ in } \overline{\Omega} \times [0, \infty) \ .$$

Then, on $\partial\Omega$, we have

$$\sigma \partial_t U + \partial_\nu U = \Big(\frac{-2\sigma\gamma - (x+\mu)\cdot\nu(x)}{2(t+1)} + \frac{\sigma \parallel x + \mu \parallel_2^2}{4(t+1)^2}\Big)U.$$

Since $p > 1 + \frac{2}{N}$, using (12) and ignoring the non-negative term $\frac{\sigma\|x+\mu\|_2^2}{4(t+1)^2}$, we can reduce the last equation to:

$$\sigma \partial_t U + \partial_\nu U \geq \Big(\frac{-\varsigma N - (x+\mu)\cdot\nu(x)}{2(t+1)}\Big)U.$$

Thanks to (13) we obtain $\mathcal{B}_\sigma(U) \geq 0$ in $N_y \times [0, \infty)$. And we have $U \geq 0$ on $\partial\Omega \setminus N_y \times (0, \infty)$. An initial data φ with $\varphi \leq U(\cdot, 0)$ in $\overline{\Omega}$ permits to conclude.

■

In the case of dimension one, we use the fact that Ω is not connected. Let us write $\Omega = \mathbb{R} \setminus [a, b]$ with $a < b$ in $\mathbb{R}$, and let V be the function defined in $\overline{\Omega} \times [0, \infty)$ by:

$$V(x,t) = \begin{cases} A(t+1)^{-\gamma} \exp \frac{-\|x+\mu_1\|_2^2}{4(t+1)} & \text{if } x \leq a \\ A(t+1)^{-\gamma} \exp \frac{-\|x+\mu_2\|_2^2}{4(t+1)} & \text{if } x \geq b \ , \end{cases}$$

with A and γ like in Corollary 4.2, μ_1 and μ_2 in $\mathbb{R}$ such that

$$-(a+\mu_1) - \varsigma \geq 0 \ ,$$

and

$$(b + \mu_2) - \varsigma \geq 0 \ .$$

As $\nu(a) = 1$ and $\nu(b) = -1$, we obtain with (12)

$$\sigma \partial_t V + \partial_\nu V \geq 0 \text{ on } (\{a\} \cup \{b\}) \times [0, \infty) \ .$$

Following the proof of Corollary 4.2, we obtain this result:

Theorem 4.3 *Under conditions (2), (3), (4), (12), $N = 1$ and $p > 3$, Problem (1) admits global positive solutions vanishing at infinity, for sufficiently small initial data φ.*

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References

[1] C. Bandle, J. von Below, and W. Reichel, Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, *Rend. Lincei Math. Appl.* **17** (2006), 35–67.

[2] C. Bandle and H.A. Levine, On the existence and the nonexistence of global solutions of reaction-diffusion equations in sectorial domains, *Trans. Am. Math. Soc.* **316** (1989), 595–622.

[3] J. von Below and C. De Coster, A Qualitative Theory for Parabolic Problems under Dynamical Boundary Conditions, *Journal of Inequalities and Applications* **5** (2000), 467–486.

[4] J. von Below and G. Pincet-Mailly, Blow Up for Reaction Diffusion Equations Under Dynamical Boundary Conditions, *Communications in Partial Differential Equations* **28** (2003), 223–247.

[5] J. von Below and G. Pincet-Mailly, Blow Up for some non linear parabolic problems with convection under dynamical boundary conditions, *Discrete and Continuous Dynamical Systems* **Supplement Volume 2007** (2007), 1031–1041.

[6] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *Journal of the Faculty of Science of the University of Tokyo* **13** (1966), 109–124.

[7] O.A. Ladyženskaya, V.A. Solonnikov, and N.N. Uraltseva, Linear and quasilinear equations of parabolic type, *Trans. of Math. Monographs* **23** (1968).

[8] H.A. Levine and Q.S. Zhang, The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values, *Proceedings of the Royal Society of Edinburgh* **130 A** (2000), 591–602.

[9] K. Mochizuki and R. Suzuki, Critical exponent and critical blow up for quasilinear parabolic equations, *Israel J. Math* **98** (1997), 141–156.